Garside structure and Dehornoy ordering of braid groups for topologist (mini-course II)

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Combinatorial Link Homology Theories, Braids, and Contact Geometry
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- II-1: Dehornoy's ordering: definition
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Part II Dehornoy's ordering

II-1: Dehornoy's ordering: definition

σ-positive word

Definition

W: word over $\{\sigma_1^{\pm 1}, \ldots, \sigma_{n-1}^{\pm 1}\}$

$$W \text{ is } \begin{cases} \sigma_i - \text{positive} \iff W \begin{array}{l} \text{contains at least one } \sigma_i \\ \text{does not contain } \sigma_1^{\pm 1}, \ldots, \sigma_{i-1}^{\pm 1}, \sigma_i^{-1}. \end{array} \\ \sigma_i - \text{negative} \iff W \begin{array}{l} \text{contains at least one } {\sigma_i}^{-1} \\ \text{does not contain } \sigma_1^{\pm 1}, \ldots, \sigma_{i-1}^{\pm 1}, \sigma_i. \end{array} \end{cases}$$

A braid $\beta \in B_n$ is

$$\begin{cases} \sigma\text{-positive} \iff \beta \text{ admits } \sigma_i\text{-positive word representatives for some } i. \\ \sigma\text{-negative} \iff \beta \text{ admits } \sigma_i\text{-positive word representatives for some } i. \end{cases}$$

(Note: β is σ-positive $\iff$ β^{-1} is σ-negative.)

σ-positive word

Examples:

- $W = \sigma_2 \sigma_3^{-1}$: σ_2-positive word

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- $W = \sigma_1\sigma_2\sigma_1^{-1}$: Neither σ-positive nor σ-negative word.

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Examples:

- $W = \sigma_2\sigma_3^{-1}$: σ_2-positive word
- $W = \sigma_1\sigma_2\sigma_1^{-1}$: Neither σ-positive nor σ-negative word.
- As a braid, $\beta = \sigma_1\sigma_2\sigma_1^{-1}$ is σ-positive:

$$\sigma_1\sigma_2\sigma_1^{-1} = \sigma_2^{-1}\sigma_1\sigma_2 \quad : \sigma_1\text{-positive word}$$

At first glance, a σ-positive/negative braid seems to be very special and it looks hard to know whether a given braid is σ-positive or not (even for 3-braids).

Quiz:
Is a braid $\sigma_1\sigma_2^3\sigma_1^{-2}\sigma_2^{-2}\sigma_1\sigma_2\sigma_1^2\sigma_2^{-7}$ σ-positive ?

Dehornoy's ordering: Algebraic definition

Theorem-Definition (Denornoy '94)

Define the relation $<_D$ of B_n by

$$\alpha <_D \beta \iff \alpha^{-1}\beta \text{ is } \sigma\text{-positive as a braid.}$$

Then $<_D$ is a left ordering of B_n (Denornoy ordering): that is, $<_D$ is a total ordering and is left-invariant relation:

$$\alpha <_D \beta \Longrightarrow \gamma\alpha <_D \gamma\beta \quad (\forall \alpha, \beta, \gamma \in B_n)$$

Corollary

1. If $\beta \neq 1$, either β or β^{-1} is σ-positive.
   $\Longrightarrow$ σ-positive/negative words are ubiquitous !!!
2. A σ-positive word represents a non-trivial braid.

Consequence of orderability

From the simple fact that B_n admits a left-ordering $<_D$ (without knowing how we defined it), we can deduce several algebraic properties of B_n

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1. B_n is torsion-free:

   For non trivial β, either $1 <_D \beta$ or $1 >_D \beta$. Then

$$\begin{cases} 1 <_D \beta & \Rightarrow 1 <_D \beta <_D \beta^2 <_D \cdots <_D \beta^i <_D \cdots \\ 1 >_D \beta & \Rightarrow 1 >_D \beta >_D \beta^2 >_D \cdots >_D \beta^i >_D \cdots \end{cases}$$

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2. The group ring $\mathbb{Z}B_n$ has no zero-divisors. (i.e. for $x, y \in \mathbb{Z}B_n$, $xy \neq 0$ if $x, y \neq 0$)

Dehornoy's ordering: example

Let us look some properties of Dehornoy's ordering

1. $1 <_D \sigma_{n-1} <_D \cdots <_D \sigma_2 <_D \sigma_1$.

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3. $1 <_D \sigma_1\sigma_2^{-1}$, but

$$1 = (\sigma_1\sigma_2\sigma_1)1(\sigma_1\sigma_2\sigma_1)^{-1} >_D (\sigma_1\sigma_2\sigma_1)(\sigma_1\sigma_2^{-1})(\sigma_1\sigma_2\sigma_1)^{-1} = \sigma_2\sigma_1^{-1}$$

so, the relation $<_D$ is not preserved under conjugacy.

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   so, the relation $<_D$ is not preserved under conjugacy.
4. If β is a positive braid (product of $\sigma_1, \ldots, \sigma_{n-1}$), then $1 <_D \beta$. In particular:

   For any $\beta \in B_n$, there exists $N \in \mathbb{Z}$ such that $\beta <_D \Delta^N$.
   (Compare Property 2. above)

Property S (Subword Property)

Theorem (Property S, Lavor '96)

For any $\beta \in B_n$, $1 <_D \beta\sigma_i\beta^{-1}$.

Property S has several consequences:

Corollary

1. $\alpha\beta <_D \alpha\sigma_i\beta$ for any $\alpha, \beta \in B_n$.
2. A band generator is $<_D$-positive: $1 <_D a_{i,j}$.
3. $<_D$ is an extension of partial ordering in the classical Garside structure:
$$\alpha \preccurlyeq \beta \Rightarrow \alpha <_D \beta$$
4. $<_D$ is an extension of partial ordering in the dual Garside structure:
$$\alpha \preccurlyeq^* \beta \Rightarrow \alpha <_D \beta$$

Well-orderedness

An important consequence of the Property S is:

Theorem (Lavar, Burckel, Dehornoy, Fromentin, I)

1. The restriction of $<_D$ on B_n^+ is a well-ordering (every non-empty set admits the $<_D$-minimal element), and the ordinal of the ordered set $(B_n^+, <_D)$ is $\omega^{\omega^{n-2}}$.
2. The restriction of $<_D$ on B_n^{+*} is a well-ordering and the ordinal of the ordered set $(B_n^{+*}, <_D)$ is $\omega^{\omega^{n-2}}$.
   (Note: $B_n^+ \subset B_n^{+*}$.)

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   (Note: $B_n^+ \subset B_n^{+*}$.)

An existence of $<_D$-minimal element seems to be useful, but at this moment, no application is known.

Open Problem

Can we compute the $<_D$-minimum element of the set of positive conjugates $\{\alpha\beta\alpha^{-1} \mid \alpha \in B_n\} \cap B_n^+$ of β ?

Short σ-positive word representatives ?

The Dehornoy ordering says that every non-trivial braid is represented by a σ-positive or a σ-negative word.

Question

Is there a "good" σ-positive/-negative word representatives ?

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Theorem (Fromentin '11)

For every braid β admits a σ-positive/negative word expression which is a quasi-geodesic:

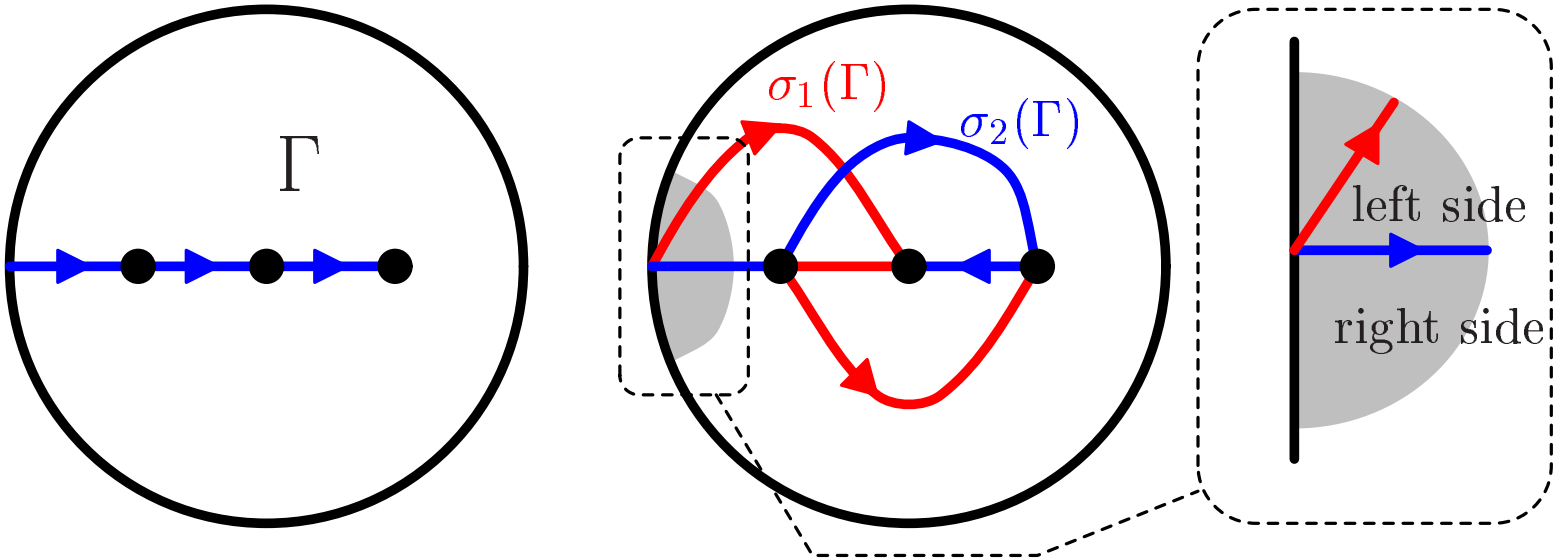
For a given n-braid β with length ℓ (with respect to $\{\sigma_1^{\pm 1}, \ldots, \sigma_{n-1}^{\pm 1}\}$), we can find a σ-positive/negative word expression of β with length at most $6(n-1)^2\ell$.

(Remark: In the proof of this theorem, dual Garside structure is effectively used !)

Dehornoy's ordering: geometric view (1)

Theorem (Fenn-Greene-Rourke-Rolfsen-Wiest '99)

$\alpha <_D \beta \iff$ $\beta(\Gamma)$ "moves the left side" of $\alpha(\Gamma)$ when we put $\beta(\Gamma)$ and $\alpha(\Gamma)$ intersect minimally

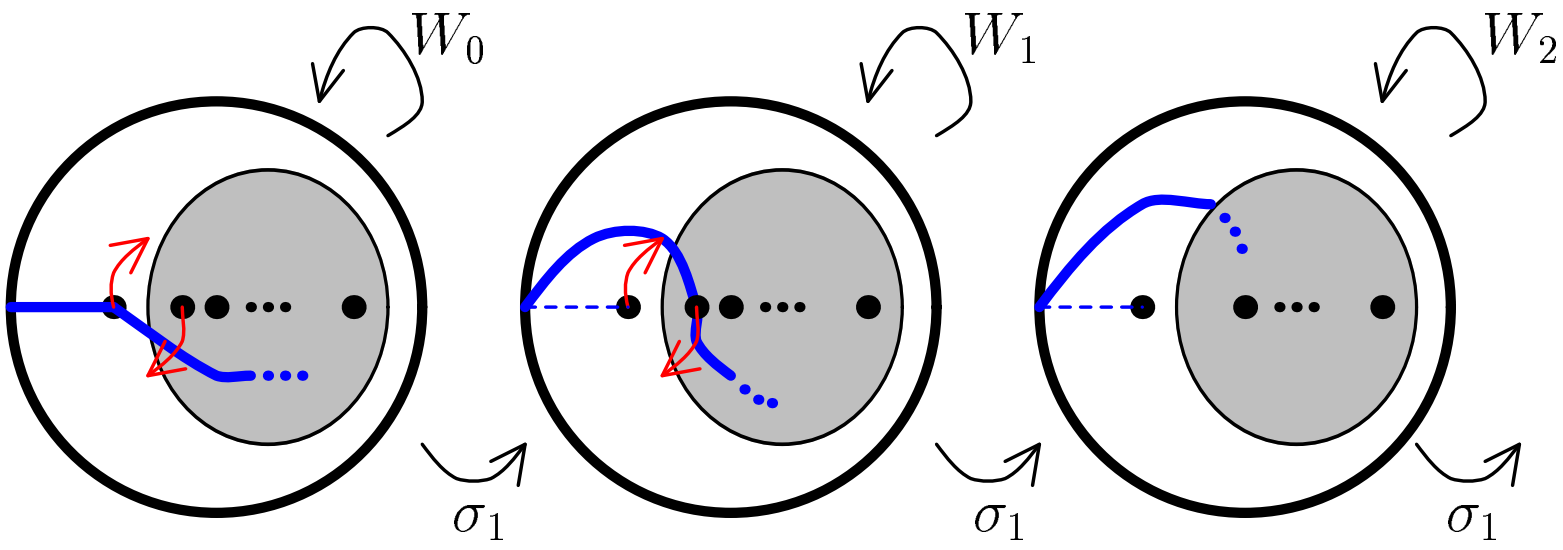


Dehornoy's ordering: geometric view (2)

[Skecth of proof:]
($\Rightarrow$) Assume β is represented by a σ_1-positive word:

$$\beta = W_n \sigma_1 \cdots W_1 \sigma_1 W_0 \quad (W_i : \text{word over } \{\sigma_2^{\pm 1}, \ldots, \sigma_{n-1}^{\pm 1}\})$$

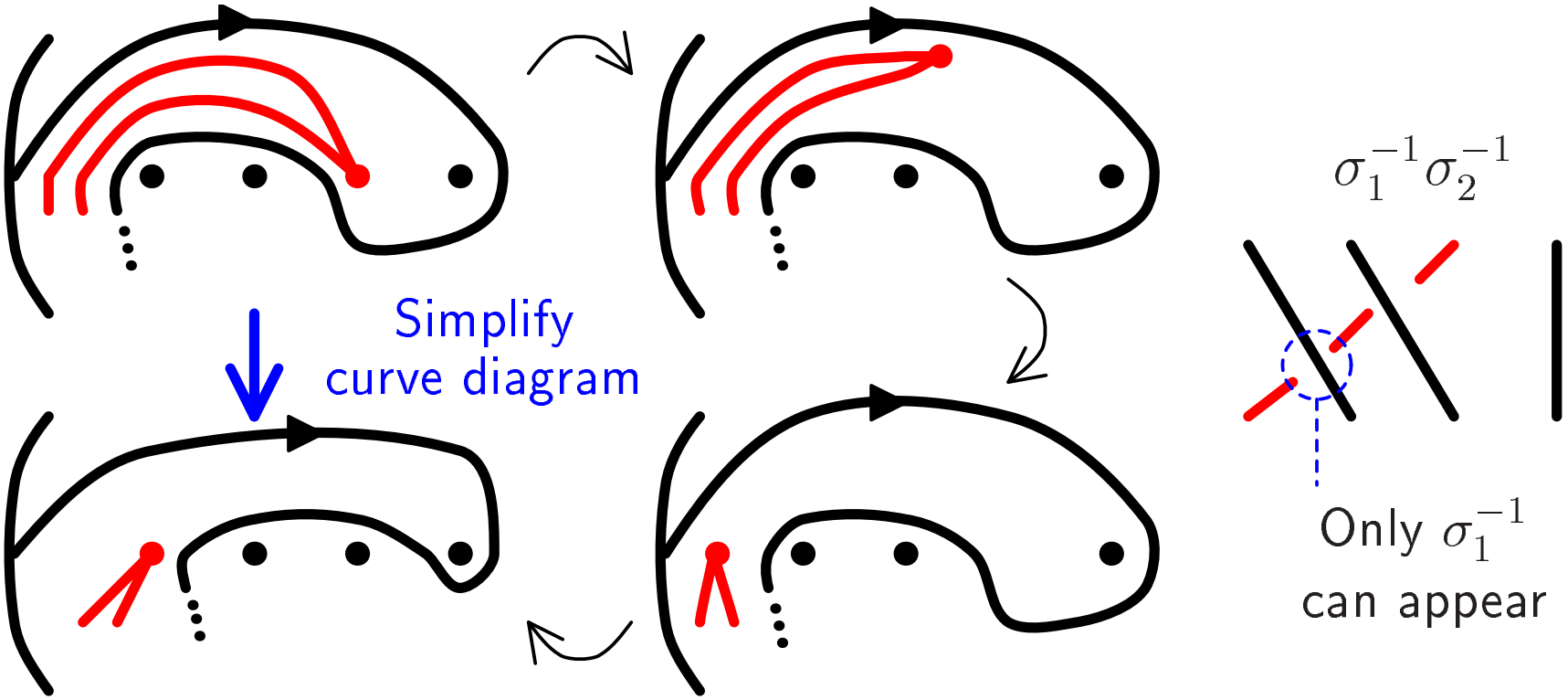
Let us look at the image of (the first segment of) Γ :



$\beta(\Gamma)$ remains to move left directions of Γ.

Dehornoy's ordering: geometric view (2)

($\Leftarrow$) If $\beta(\Gamma)$ moves the left direcition of Γ, we can simplify $\beta(\Gamma)$ by applying braid containing no σ_1:



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Geometric point of view yields:

- A simple proof of Property S.

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Important prospect

A reasonable procedure of simplifying curve diagrams yields a connection of topology/geometry of braids and algebraic structure (Garside normal form, Dehornoy ordering) of braid groups.

A curve diagram is a deep and important object
than our first impression (although it is very simple) !!!

(There might be other nice property of braids read from curve diagrams...)

Dehornoy's ordering: More schematic geometric view

We want to remove "up to isotopy" in curve diagram definition of $<_D$.

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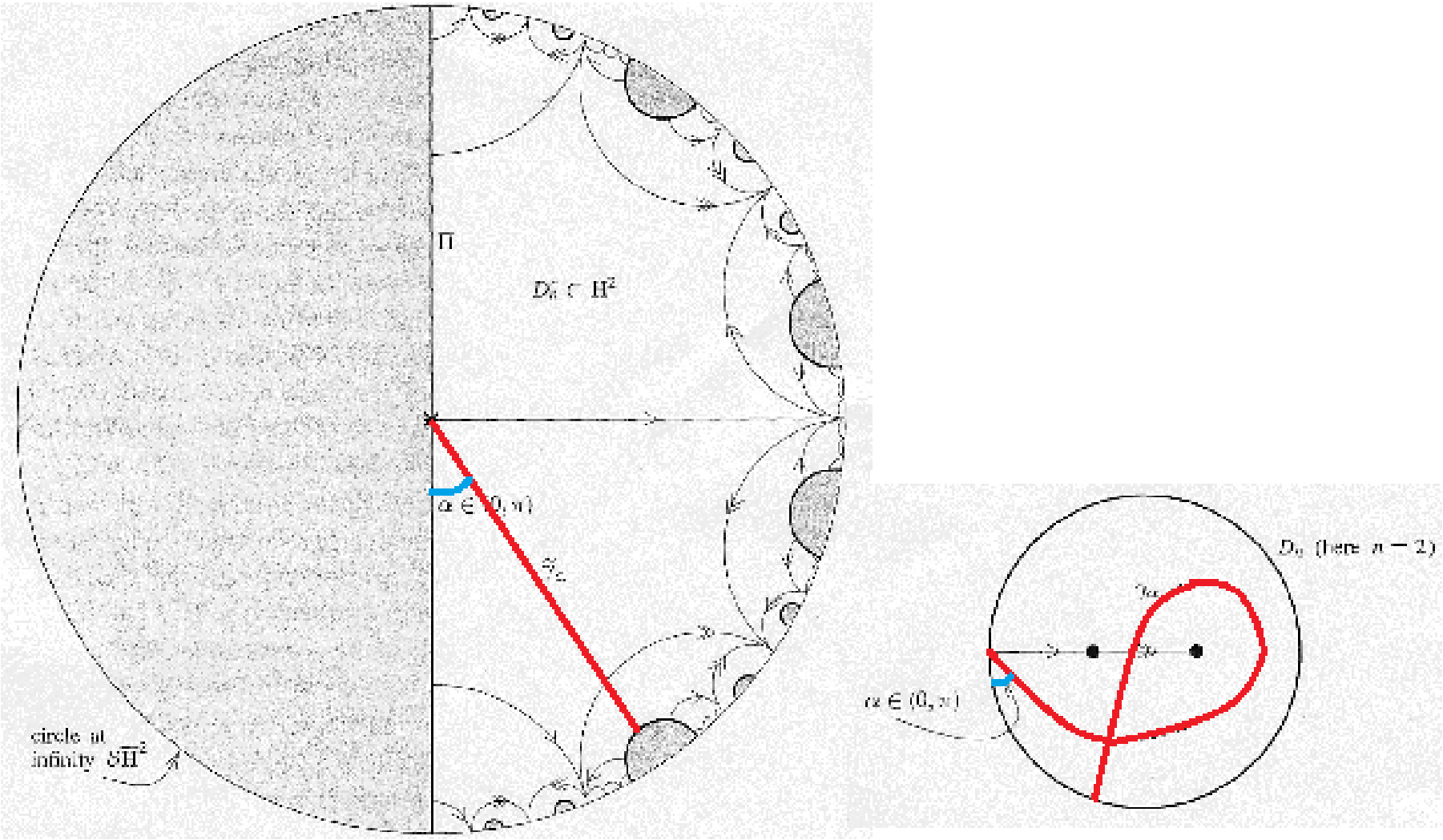
Solution

The hyperbolic geometry is useful to give the " best" representative of curves: Equip hyperbolic structure of D_n. Then,

- By isotopy, one can realize every non-trivial curve as geodesic.
- Two geodesics on hyperbolic surface minimally intersect.
- In the universal covering of $D_n \subset \mathbb{H}^2$, geodesic is easy to see: if we put the base point in the center of disc model of $\mathbb{H}^2$, geodesic is just a straight line.

Dehornoy's ordering: More schematic geometric view



(Figure borrowed from Short-Wiest's paper)

Nielsen-Thurston type ordering

Using hyperbolic geometry construction, we can generalize the Dehornoy ordering for mapping class group of surface with non-empty boundary:

S: Hyperbolic Surface with non-empty geodesic boundary
$\mathbb{H}^2 \supset \widetilde{S} \xrightarrow{\pi} S$: Universal covering

By considering the lifted action on boundary at infinity (which does not depend on a choice of representative homeomorphism) we get an injective homomorphism

$$\Theta : MCG(S) \to \mathrm{Homeo}^+(\mathbb{R})$$

called the Nielsen-Thurston map.

Remark

The map Θ is not canonical – it may depends on various intermediate choices (hyperbolic metric etc...)

Nielsen-Thurston type ordering

Definition

Take an ordered, countable dense subset $\{x_1, x_2, \ldots\}$ of $\mathbb{R}$. For $\phi, \psi \in MCG(S)$, define

$$\phi < \psi \iff \exists j \ s.t. \begin{cases} [\Theta(\phi)](x_i) = [\Theta(\psi)](x_i) & i = 1, \ldots, j-1 \\ [\Theta(\phi)](x_j) < [\Theta(\psi)](x_j) \end{cases}$$

This defines a left ordering of $MCG(S)$, called the Nielsen-Thurston type orderings.

Remark

The Dehornoy ordering is regarded as a special one of the Nielsen-Thurston type ordering.

II -2 Technique to compute Dehornoy ordering (handle reduction)

Handle reduction

How to determine $1 <_D \beta$ or not ?

Observation

$\sigma_1\sigma_2\sigma_1^{-1}$ is not σ-positive word, but we can rewrite

$$\sigma_1\sigma_2\sigma_1^{-1} = \sigma_2^{-1}\sigma_1\sigma_2 \quad (\sigma_1\text{-positive word})$$

More generally, $\sigma_1\sigma_2^k\sigma_1^{-1} = \sigma_2^{-1}\sigma_1^k\sigma_2$

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<u>Idea:</u>

By modifying the word of the form $\sigma_1(\sigma_1\text{-free word})\sigma_1^{-1}$ (this is a bad sequence) we may get σ_1-positive/negative word.

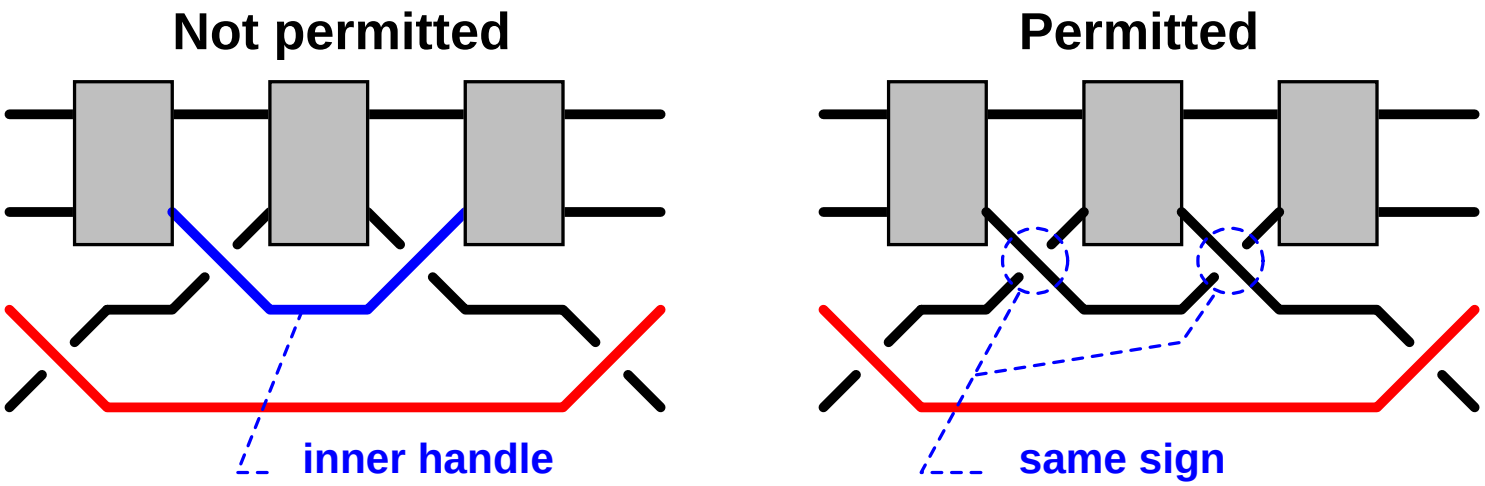
Handle reduction

Definition

A permitted handle of a braid word w is a subword of the form

$$h = \sigma_1^{\pm 1} V_0 \sigma_2^{\varepsilon} V_1 \sigma_2^{\varepsilon} \cdots \sigma_2^{\varepsilon} V_k \sigma_1^{\mp 1}$$

where $\varepsilon \in \{\pm 1\}$ and V_i is a word containing no $\sigma_1^{\pm 1}, \sigma_2^{\pm 1}$.



Handle reduction

Definition

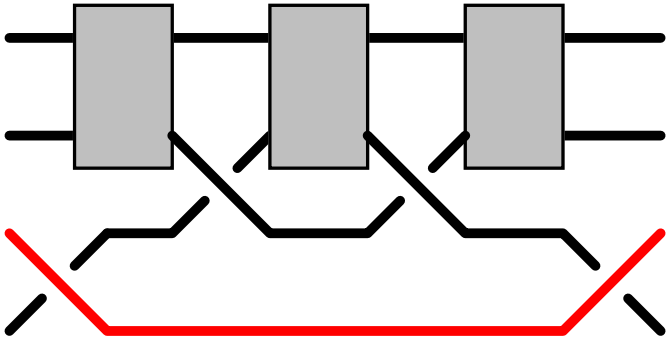
The handle reduction of a permitted handle h in a braid word w is replacement

$$h = \sigma_1^{\pm 1} V_0 \sigma_2^{\varepsilon} V_1 \sigma_2^{\varepsilon} \cdots \sigma_2^{\varepsilon} V_k \sigma_1^{\mp 1}$$

with

$$red(h) = V_0 (\sigma_2^{\mp 1} \sigma_1^{\varepsilon} \sigma_2^{\pm 1}) V_1 \cdots (\sigma_2^{\mp 1} \sigma_1^{\varepsilon} \sigma_2^{\pm 1}) V_k$$

permitted handle



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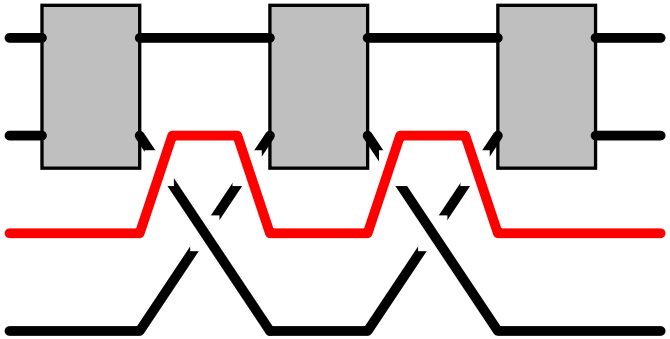

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- Nevertheless, handle reduction eventually yields a σ-positive or σ-negative word:

Theorem (Dehornoy '97)

For a given n-braid word of length ℓ, after at most $2^{n^4\ell}$ (exponential) times of handle reductions, we arrive at a σ-positive or σ-negative word.

The proof is not so simple, because we have no good notion of complexity which decrease by applying handle reduction.

Example of handle reduction

Let us use handle reduction to find σ-positive or σ-negative word for

$$\sigma_1\sigma_2\sigma_3\sigma_2\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}.$$

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Find a handle:

$$\underline{\sigma_1\sigma_2\sigma_3\sigma_2\sigma_1^{-1}}\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}.$$

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Handle reduction (get longer word!):

$$\sigma_2^{-1}\sigma_1\sigma_2\sigma_3\sigma_2^{-1}\sigma_1\sigma_2\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}.$$

Example of handle reduction

Let us search next handle for

$$\sigma_2^{-1}\sigma_1\sigma_2\sigma_3\sigma_2^{-1}\sigma_1\sigma_2\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}.$$

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By looking for the pattern $\sigma_1 \cdots \sigma_1^{-1}$, we find:

$$\sigma_2^{-1}\sigma_1\sigma_2\sigma_3\sigma_2^{-1}\underline{\sigma_1\sigma_2\sigma_2^{-1}\sigma_1^{-1}}\sigma_2^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}.$$

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This handle is not permitted – we look for inner handles

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and do handle reduction (this is just a cancellation):

$$\sigma_2^{-1}\sigma_1\sigma_2\sigma_3\sigma_2^{-1}\sigma_1\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}.$$

Example of handle reduction

Handel reduction again:

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Example of handle reduction

Handel reduction again:

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$$\sigma_2^{-1}\sigma_3^{-1}\sigma_3^{-1}.$$

We eventually find σ-negative word (so, word without handles).

Handle reduction

Experimental fact

Among known algorithms to compute the Dehornoy ordering, a handle reduction method is the best:

- It is easy to do (even by hands and to implement computor program).
- It converges in linear time (and will produce a short σ-positive/negative representatives).

Handle reduction

Experimental fact

Among known algorithms to compute the Dehornoy ordering, a handle reduction method is the best:

- It is easy to do (even by hands and to implement computor program).
- It converges in linear time (and will produce a short σ-positive/negative representatives).

Note that in the previous theorem only gives an exponential upper bound $2^{4^n \ell}$. This suggests our current understanding of handle reduction is very poor .

Handle reduction

Question

1. Prove handle reduction converges very fast (conjectually in linear time, but polynomial time bound is still interesting)
2. Give a topological/geometric prospect of handle reduction. What is handle "reduction" reducing ?
3. Generalize a theory of handle reduction technique for other groups. (Note: handle reduction can be seen as standard reducing operation $xx^{-1} \mapsto \varepsilon$, which is basic in the free group. There might be a good notion and properties of "handle-reduced" words in more general group.)

A handle reduction seems to reflect unknown combinatorics and prospects in braid groups...

II-2 Application to (contact) topology (1): Knot theory

The Dehornoy ordering

The Dehornoy ordering $<_D$ is fundamental, but quite interesting object related to various aspects of the braid groups:

- Combinatorics (σ-positive words)
- Topology (Curve diagram)
- Geometry (Hyperbolic geometry, Nielsen-Thruston action)

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- Set-theory (distribuitivive operations on sets)
- Surface triangulation (Dynnikov coordinate, Mosher's normal form of MCG)
- Possibly more unknown prospects.....

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- Set-theory (distribuitivive operations on sets)
- Surface triangulation (Dynnikov coordinate, Mosher's normal form of MCG)
- Possibly more unknown prospects.....

Moreover, in a theory of left-ordering of groups the Dehornoy ordering is a source of various important examples.

The Dehornoy ordering

Natural question is:

Question

Can we use Dehornoy ordering to study topology/geometry ?

Naive speculation

The Dehornoy ordering $<_D$ can be seen as a complexity of braids.
$\Rightarrow <_D$ may also be regarded as a complexity of geometric object (knots and links, for example) arising from braids.

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Naive speculation

The Dehornoy ordering $<_D$ can be seen as a complexity of braids.
$\Rightarrow$ $<_D$ may also be regarded as a complexity of geometric object (knots and links, for example) arising from braids.

Surprisingly, this speculation is true, and

Conclusion

If K is a closure of a braid β which is sufficiently complicated (with respect to $<_D$), then property of K is directly read from β.

The Dehornoy floor of braids

Definition

The Dehornoy floor of braid β is an integer $[\beta]_D$ satisfying

$$\Delta^{2[\beta_D]} \leq_D \beta <_D \Delta^{2[\beta_D]+2}$$

The Dehornoy floor is regarded as a numerical complecity of braids measured by the Dehornoy ordering.

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Lemma

1. The Dehornoy floor map $[\,]_D : B_n \to \mathbb{Z}$ is a quasi-morphism of defect one:
$$|[\alpha\beta]_D - [\alpha]_D - [\beta]_D| \leq 1$$
2. If α and β are conjugate, $|[\alpha]_D - [\beta]_D| \leq 1$.

The Dehornoy floor of braids

Proposition

If the closure of an n-braid β admits destabilization (i.e. β is conjugate to $\alpha\sigma_n^{\pm 1}$ for $\alpha \in B_{n-1}$), then $|[\beta]_D| \leq 1$.

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Proposition

If the closure of an n-braid β admits destabilization (i.e. β is conjugate to $\alpha\sigma_n^{\pm 1}$ for $\alpha \in B_{n-1}$), then $|[\beta]_D| \leq 1$.

Proof:
Assume β is conjugate to $\alpha\sigma_{n-1}^{\pm 1}$, so is conjugate to

$$\beta' = \Delta(\alpha\sigma_{n-1}^{\pm 1})\Delta^{-1} = \{\text{word over } \sigma_2^{\pm 1}, \ldots, \sigma_{n-1}\}\sigma_1^{\pm 1}.$$

Then,

$$\Delta^{\pm 2}\beta' = \{\text{word over } \sigma_2^{\pm 1}, \ldots, \sigma_{n-1}\} \cdot \underbrace{(\Delta^{\pm 2}\sigma_1^{\pm 1})}_{\sigma_1-\text{positive/negative}}$$

So

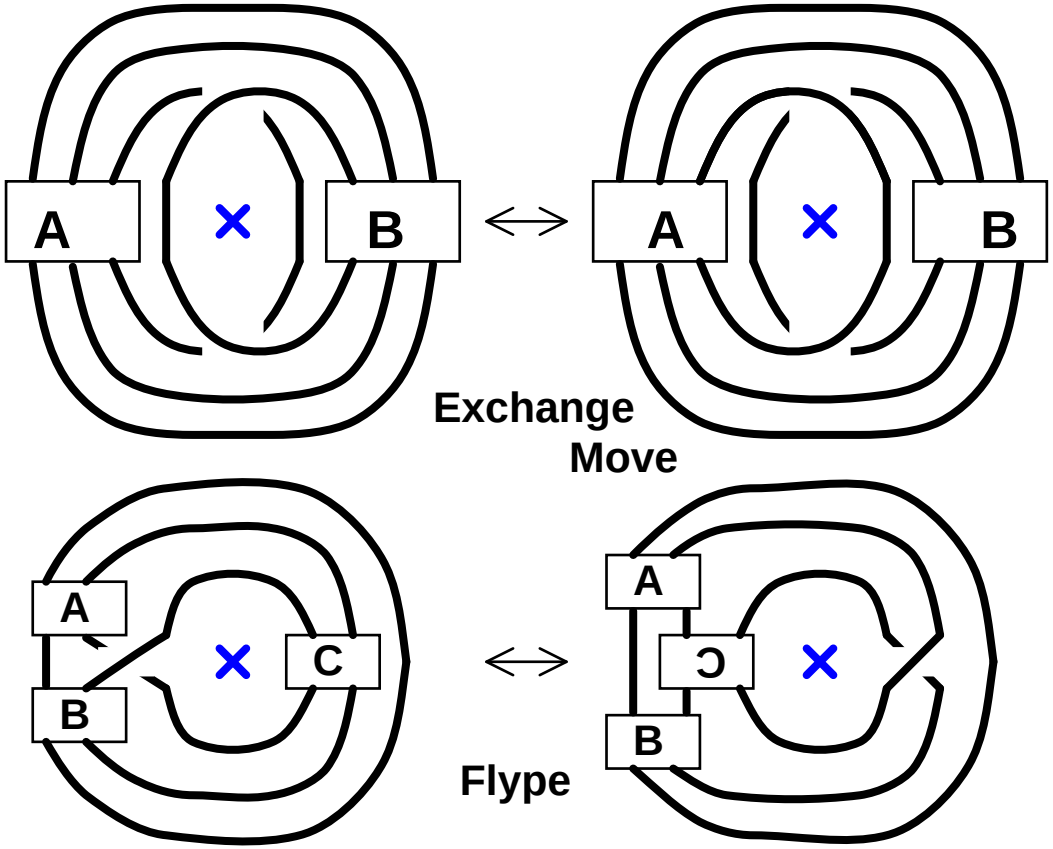
$$\Delta^{-2} <_D \beta' <_D \Delta^2.$$

The Dehornoy floor of braids

By the same argument,

Proposition

1. If the closure of an n-braid β admits exchange move then $|[\beta]_D| \leq 1$.
2. If the closure of an n-braid β admits flype then $|[\beta]_D| \leq 2$.



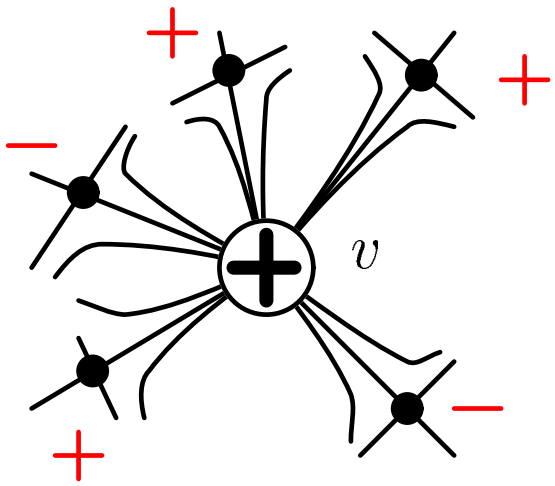
The Dehornoy floor of braids

These "template moves" geometrically appears in the theory of Birman-Menasco's braid foliation theory (cf. LaFountain's lecture) and the Dehornoy ordering is related to the braid foliation.

Key Proposition

$L = \widehat{\beta}$:Closed braid
$F \in S^3 - L$: Seifert/incompressible closed surface.

If the braid foliation of F has a positive vertex v with p positive saddles and n negative saddles around v, then $-n \leq [\beta]_D \leq p$.



3 positive saddles

2 negative saddles

$\Rightarrow -2 \leq [\beta]_D \leq 3$

Dehornoy floor and knot theory

By using the braid foliation theory, we have several close connections between the Dehornoy's ordering (recall it is defined by algebraic way !!) and knot theory:

Proposition (Malyutin-Netsvetaev'04, I.)

1. If $|[\beta]_D| > 1$, then $\widehat{\beta}$ is prime, non-split, non-trivial link.
2. For $n \in \{2, 3, \ldots, \}$ there exists a number $r(n) \in \mathbb{Z}$ such that: The closure of two n-braids α and β with $|[\alpha]_D|, |[\beta]_D| \geq r(n)$ represent the same link if and only if α and β are conjugate. (Moreover, the braid index of $\widehat{\alpha} = n$).

Surprising consequence

If the Dehornoy floor is sufficiently large,

Algebraic link problem = conjugacy problem of braids!!

Dehornoy floor and knot theory

More direct connections for Dehornoy ordering and knots:

Theorem (I, '12)

1. Let $\beta \in B_n$. If $g(\widehat{\beta})$ the genus of a knot $\widehat{\beta}$,

$$|[\beta]_D| \leq \frac{4g(K)}{n+2} - \frac{2}{n+2} + \frac{3}{2} \leq g(K) + 1.$$

   Thus, a complicated braid (with respect to the Dehornoy ordering) yields a complicated knot (with respect to topology – Thurston norm)

2. Assume that $|[\beta]_D| \geq 2$. Then,

$$\widehat{\beta} \text{ is a } \begin{cases} \text{torus knot} \\ \text{satellite knot} \\ \text{hyperbolic knot} \end{cases} \iff \beta \text{ is } \begin{cases} \text{periodic} \\ \text{reducible} \\ \text{pseudo-Anosov} \end{cases}$$

Further application: quantum invariants

Recall the definitions of quantum invariants:
$U_q(\mathfrak{g})$: Quantum enveloping algebra of semi-simple Lie algebra $\mathfrak{g}$
V: $U_q(\mathfrak{g})$-module(s)

$$\begin{tikzcd}
\{\text{Braids}\} \arrow[r, "\textit{Closure}"] \arrow[d, "\rho_V"', "\textit{Quantum representation}"] & \{\text{(Oriented) Links }\} \arrow[r, "\textit{Surgery}"] \arrow[d, "\textit{Quantum invariant}"] & \{\text{Closed 3-manifolds}\} \arrow[d, "\textit{Quantum invariant}"] \\
GL(V^{\otimes n}) \arrow[r, "\text{“}\textit{Trace}\text{''}"] & \mathbb{C}[q, q^{-1}] \arrow[r, "q=e^{\frac{2\pi\sqrt{-1}}{N}}", "\text{Take linear sums}"'] & \mathbb{C}
\end{tikzcd}$$

$\rho_V : B_n \to \mathrm{GL}(V)$ is called quantum representation.

Big open problem in knot theory

Open problem

Which quantum invariants detect the unknot ?
Does Jones polynomial ("the simplest" quantum invariant) detect the unknot ?

Big open problem in knot theory

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Which quantum invariants detect the unknot ?
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From the construction of quantum invariants, we have

Observations (Bigelow)

If an n-braid $\alpha \in Ker\rho_V$, then for any $\beta \in B_n$

$$Q_V(\widehat{\alpha\beta}) = Q_V(\widehat{\beta})$$

$\Rightarrow$ Quantum representation ρ_V is not faithful, then Q_V is not strong – it fails to detect the unknot.

Is it true? The link $\widehat{\alpha\beta}$ may be the same as $\widehat{\beta}$...

Closed braids via normal subgroups

Bigelow's speculation is true:

Theorem (I.)

Let N be the non-trivial, non-central normal subgroup of B_n. Then for any $\beta \in B_n$, the set of knots (and links)

$$\{\widehat{\alpha\beta} \mid \alpha \in N\}$$

contains infinitely may distinct (hyperbolic) knots. (i.e. normal subgroup of B_n produces infinitely many knots.)

It sounds "obvious", but how to prove ?

- We cannot use (easy-to-calculate) invariant to distinguish knots !!!
- We do not know an element in N explicitly.

Consequences

One can attack faithfulness of knot invariants via braid group representations:

Corollary (I.)

1. If quantum representations $\rho_i : B_n \to \mathrm{GL}(V_i^{\otimes n})$ $(i = 1, \ldots, k)$ are not faithful, for any knot type K, there exists infinitely many mutually different, (hyperbolic) knot $K_1, K_2, \ldots$ such that

$$Q_{V_i}(K) = Q_{V_i}(K_*) \quad (* = 1, 2, \ldots) \text{ for all } i = 1, \ldots, k.$$

2. (Bigelow) If the 4-strand (reduced) Burau representation

$$\rho_4 : B_4 \to \mathrm{GL}(3, \mathbb{Z}[q^{\pm 1}])$$

is not faithful, then there exists a non-trivial knot with trivial Jones polynomial.

Proof of Theorem

Surprisingly, theorem is a consequence of a purely algebraic statement for the Dehornoy ordering.

Theorem′ (I.)

A non-trivial normal subgroup N of B_n is unbounded with respect to the Dehornoy ordering $<_D$: For any $\beta \in B_n$, there exists $\alpha \in N$ such that $\alpha^{-1} <_D \beta <_D \alpha$.

<u>Theorem′ $\Rightarrow$ Theorem</u>

- If N is unbounded, so is the set $\{\beta\alpha \mid \alpha \in N\}$ for any β. (Moreover, it contains infinitely many pseudo-Anosov elements).
- Inequality of the Dehornoy floor and knot genus (previous theorem) shows the set of knots
$$\{\widehat{\beta\alpha} \mid \alpha \in N\}$$
is infinite (because it contains arbitrary large genus knot.)

II-4: Application (2): FDTC and contact geometry

Fractional Dehn twist coefficient

S: surface with non-empty boundary
$C \subset \partial S$: connected component of ∂S.

Using Nielsen-Thurston theory, Honda-Kazez-Matić defined the fractional Dehn twist coefficients (FDTC) (with respect to C) $c(\phi, C) \in \mathbb{Q}$:

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<u>1. Periodic case:</u>

Take $N > 0$ so that $\phi^N = T_C^M \cdots$ (Dehn twsits along other boundaries). Then,

$$c(\phi, C) = \frac{M}{N}$$

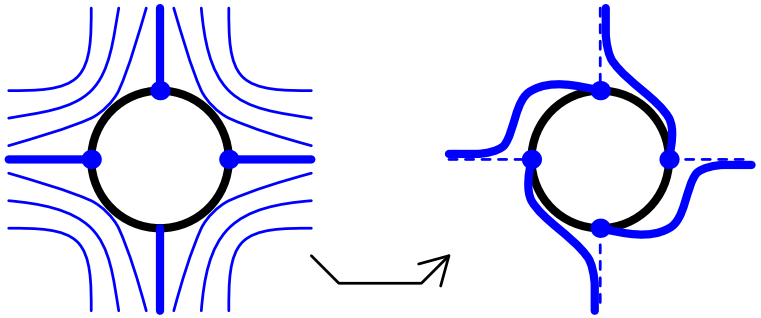
(we regard ϕ is rotation by $\frac{2\pi M}{N}$ near C)

Fractional Dehn twist coefficient

2. Pseudo-Anosov case:

Consider a pseudo-Anosov homormophism representative. Using the singular leaves of its invariant foliation near C, in the neighborhood of C we identify ϕ with the rotation by $\frac{2\pi M}{N}$, and define

$$c(\phi, C) = \frac{M}{N}$$



3. Reducible case:

Consider irreducible component of ϕ containing C.

Alternative definition of FDTC

Let us use a Nielsen-Thurston map

$$\Theta : MCG(S) \to \mathrm{Homeo}^+(\mathbb{R}).$$

T_C is central and we may normalize Θ so that $\Theta(T_C) : x \mapsto x+1$: i.e.,

$$\Theta : MCG(S) \to \widetilde{\mathrm{Homeo}}^+(S^1)$$

(Here $\widetilde{\mathrm{Homeo}}^+(S^1) = \{\widetilde{f} : \mathbb{R} \to \mathbb{R} \mid f : \mathbb{R}/\mathbb{Z} = S^1 \to S^1\}$). Consider the translation number

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Theorem (I-Kawamuro)

$$c(\phi, C) = \tau \circ \Theta(\phi).$$

Relation to the Dehornoy floor

Let us consider a normalized Nielsen-Thurston map for braids,

$$\Theta : B_n \to \widetilde{\mathrm{Homeo}}^+(S^1)$$

$\Theta(\Delta^2) : x \mapsto x+1$. Moreover, we may further arrange so that

$$\alpha <_D \beta \iff [\Theta(\alpha)](0) < [\Theta(\beta)](0).$$

Then, the translation number is nothing but the "stable" Dehornoy floor:

Theorem (I-Kawamuro)

$$c(\phi, C) = \lim_{N\to\infty} \frac{[\beta^N]_D}{N}$$

Application to contact geometry

Summary

FDTC = generalization of the Dehornoy floor

In particular, we have:

Application to contact geometry

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In particular, we have:

- Fast computation of FDTC without knowing Nielsen-Thurston type.

  (For the braid group, we have fast (conjectured to be linear time) algorithm to compute the Dehornoy ordering.)

Application to contact geometry

Summary

FDTC = generalization of the Dehornoy floor

In particular, we have:

- Fast computation of FDTC without knowing Nielsen-Thurston type.

  (For the braid group, we have fast (conjectured to be linear time) algorithm to compute the Dehornoy ordering.)
- Relationship between Nielsen-Thurston ordering of Mapping class groups and contact 3-manifolds.

Application to contact geometry

Viewing FDTC as generalization of Dehornoy floor (and the theory of Open book foliation) allows us to generalize theorem for Dehornoy ordering and knot theory for FDTC and (contact) 3-manifolds.

Theorem (I-Kawamuro)

Let (S, ϕ) be an open book decomposition of (contact) 3-manifold (M, ξ).

1. If $|c(\phi, C)| \geq 4$ for all boundary of S, then

$$M \text{ is a } \begin{cases} \text{Seifert-fibered manifold} \\ \text{toroidal manifold} \\ \text{hyperbolic manifold} \end{cases} \iff \phi \text{ is } \begin{cases} \text{periodic} \\ \text{reducible} \\ \text{pseudo-Anosov} \end{cases}$$

2. If S is planar and $c(\phi, C) > 1$ for all boundary of S, then ξ is a tight contact structure on M.

II-5: The Dehornoy ordering and Garside theory

Garside theory technique to compute Dehornoy's ordering: Alternating normal form

Question

Can we use Garside structure to study/compute normal form ?

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At first glance:

- There seems to be little connection between Garside normal form and Dehornoy ordering – normal form is far from σ-positive word.
- However, $<_D$ is an extension of subword ordering $\preccurlyeq$ in Garside theory.

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- However, $<_D$ is an extension of subword ordering $\preccurlyeq$ in Garside theory.

Natural speculation

For B_n^+, using normal forms we can extend $\preccurlyeq$ to get the Dehornoy ordering $<_D$.

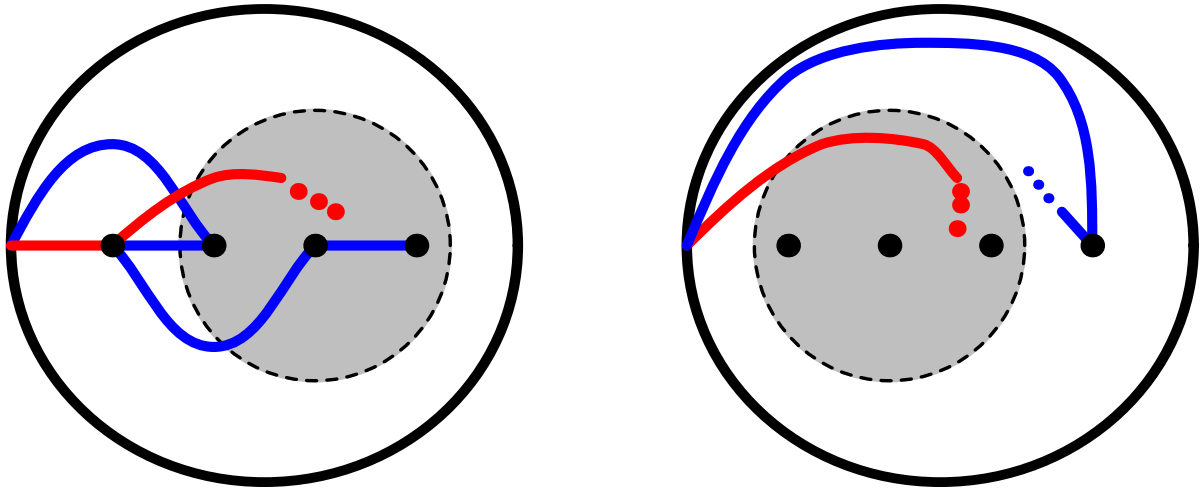
Geometric speculation

From the curve diagram interpretation...

$$(\text{Braids without } \sigma_1) <_D \sigma_1$$

$$(\text{Braids without } \sigma_{n-1})(\text{Braids without } \sigma_1) <_D \sigma_{n-1} \cdots \sigma_2 \sigma_1$$

Chasing the patterns of appearance of σ_1 and σ_{n-1} can estimate the Dehornoy ordering.



Garside theory technique to compute Dehornoy's ordering: Alternating normal form

To capture the patterns of σ_1 and σ_{n-1}, we use Garside structure idea: Let us define

$$\begin{cases} A = \{\text{Positive words over } \sigma_1, \ldots, \sigma_{n-2}\} \subset B_n^+ \\ B = \{\text{Positive words over } \sigma_2,\ dots, \sigma_{n-1}\} \subset B_n^+ \end{cases}$$

and

$$\begin{cases} \Delta_A = (\sigma_1\sigma_2 \cdots \sigma_{n-2})(\sigma_1\sigma_2 \cdots \sigma_{n-3}) \cdots (\sigma_1\sigma_2)(\sigma_1) \in A \\ \Delta_B = (\sigma_2\sigma_3 \cdots \sigma_{n-1})(\sigma_2\sigma_3 \cdots \sigma_{n-2}) \cdots (\sigma_2\sigma_3)(\sigma_2) \in B \end{cases}$$

(Both A and B are isomorphic to B_{n-1}^+.)

Garside theory technique to compute Dehornoy's ordering: Alternating normal form

Recall that in the normal form, we decompose β as a product of simple elements by repeatedly computing $\beta \wedge \Delta$.
WE replace the role of Δ by A and B.

Definition

For $\beta \in B_n^+$, define

$$\begin{aligned} \beta \wedge A &= \max_{\preccurlyeq, N>0} \beta \wedge \Delta_A^N \in A, \\ &= \max_{\preccurlyeq} \{\alpha \in A \mid \beta\alpha^{-1} \in A\}. \end{aligned}$$

$\beta \wedge B$ is defined similarly.

Alternating normal form

Definition

For $\beta \in B_n^+$, the alternating decomposition is a factorizatino of β

$$\mathcal{A}(\beta) = b_k a_k \cdots b_1 a_1 b_0$$

where $a_i \in A$, $b_i \in B$ is defined by:

$$\begin{cases} b_0 = \beta \wedge B \\ a_i = \beta(b_{i-1} \cdots b_1 a_1 b_0)^{-1} \wedge A \\ b_i = \beta(a_i \cdots b_1 a_1 b_0)^{-1} \wedge B \end{cases}$$

Alternating normal form

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For $\beta \in B_n^+$, the alternating decomposition is a factorizatino of β

$$\mathcal{A}(\beta) = b_k a_k \cdots b_1 a_1 b_0$$

where $a_i \in A$, $b_i \in B$ is defined by:

$$\begin{cases} b_0 = \beta \wedge B \\ a_i = \beta(b_{i-1} \cdots b_1 a_1 b_0)^{-1} \wedge A \\ b_i = \beta(a_i \cdots b_1 a_1 b_0)^{-1} \wedge B \end{cases}$$

Example

For $\beta = (\sigma_1\sigma_2)^3 = \sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2$,

$$\mathcal{A}(\beta) = \sigma_1 \cdot \sigma_2^2 \cdot \sigma_1 \cdot \sigma_2^2$$

Alternating normal form

Note we have an injective homomorphism

$$\Phi : B \to A \cong B_{n-1}^{+} \subset B_n^{+}, \quad \Phi(\beta) = \Delta\beta\Delta^{-1}$$

Theorem (Dehornoy '09, I. '10)

For two positive braids α and β, let

$$\begin{cases} \mathcal{A}(\alpha) = b_k a_k \cdots b_1 a_1 b_0 \\ \mathcal{A}(\beta) = B_{k'} A_{k'} \cdots B_1 A_1 B_0 \end{cases}$$

be their alternating decomposition. Then

$$\alpha <_D \beta \iff \begin{cases} k < k' \text{ or,} \\ k = k' \text{ and } \exists i \text{ such that} \\ B_k = b_k, A_k = a_k, \cdots A_{i+1} = a_{i+1}. \\ \Phi(B_i) <_D \Phi(b_i) \text{ or }, \Phi(B_i) = \Phi(b_i) \text{ and } A_i <_D a_i \end{cases}$$

Alternating normal form

Theorem (Dehornoy '09, I. '10)

That is, sequence of $(n-1)$ braids coming from alternating decomposition of β

$$(,\ldots,\Phi(B_{k'}),A_{k'},\ldots,\Phi(B_1),A_1,\Phi(B_0))$$

is larger than the sequence from α

$$(,\ldots,\Phi(b_k),a_k,\ldots,\Phi(b_1),a_0,\Phi(b_0))$$

with respect to the lexicographical ordering based on the Dehornoy ordering $<_D$ (of B_{n-1}).

Thus, by using Garside theory method, we can reduce the computation of the Dehornoy ordering of B_n to the computation in B_{n-1}.

Application

Like normal forms, alternating decomposition is easy to calculate:

Proposition (Dehornoy)

The alternating decomposition of positive braid of length ℓ is computed in time $\mathcal{O}(\ell^2)$. In particular, for given (not necessarily positive) braid β of length ℓ, whether $1 <_D \beta$ or not is determined in time $\mathcal{O}(\ell^2)$.

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Iteration of alternating decomposition provides nice enumeration of positive braids with respect to $<_D$, and allows us to interpret $<_D$ as lexicographical ordering:

Corollary (Burckel '97, I. '10)

The well-ordered set $(B_n^+, <_D)$ is of type $\omega^{\omega^{n-2}}$.

Remarks

1. Similar arguments apply for the dual Garside structure (Fromentin, I.)

   In the classical case, we used

   $$A = B_{n-1}^{+} \text{ and } B = \Delta(B_{n-1}^{+})\Delta^{-1}.$$

   In the dual case, we use

   $$A_1 = B_{n-1}^{+*}, A_2 = \delta B_{n-1}^{+*}\delta^{-1}, A_3 = \delta^2 B_{n-1}^{+*}\delta^{-2}, \ldots$$

2. The Dehornoy ordering is a speicial one of the Nielsen-Thurston type ordering. Using the variant of alternating normal form, we have a similar result for a special class of Nielsen-Thurston type ordering called of finite type.

   Note: For Nielsen-Thurston type orderings, we can not use σ-positive words – now, the Garside structure provides algebraic, combinatorial description of geometrically defined orderings.

Further readings

For basics of Garside normal forms, Section 1 of

- J. Birman, V. Gebhardt, and J. González-Meneses ,*Conjugacy in Garside group I: cycling, powers and rigidity*, Groups Geom. Dyn **1** (2007) 221-279.

or, Section 5 of

- J. Birman, and T. Brendle, *Braids: A survey*, Handbook of knot theory, 19–103.

contains a nice and concise overview.

Further readings

For the basics of the Dehornoy ordering,

- P. Dehornoy, I.Dynnikov, D.Rolfsen and B.Wiest, *Ordering Braids,* Mathematical Surveys and Monographs **148**, Amer. Math. Soc. 2008.

Also, a survey

- P. Dehornoy *Braid order, sets, and knots.* Introductory lectures on knot theory, 77-96, Ser. Knots Everything, 46, World Sci. Publ., Hackensack, NJ, 2012.

contains brief explanation of applications to knot theory.